Math 275D Lecture 15 Notes

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1 Orthogonal Polynomial Method for Constructing Brownian Motion

1.1 Overview

This lecture, we will talk about a method used to construct Brownian motion which is good for computers to simulate. Previously, we constructed Brownian motion on dyadic rational numbers and extended it continuously to $\mathbb{R}$. We also constructed Brownian motion as a limit in distribution of scaled simple random walks using Donsker's theorem.

Here are the basic properties of Brownian motion:

1. $B(0) = 0$.
2. $B(t) \sim N(0, t)$.
3. $B(I_1) \perp B(I_2)$ if $I_1 \cap I_2 = \varnothing$, where $B(I) = B(b) - B(a)$ if $I = [a, b]$.
4. $B(\cdot) \in C([0, 1])$.

We will construct a sequence of random functions $f^{(n)}$ that converges to Brownian motion in the $\| \cdot \|_\infty$ sense. The limiting random function will satisfy the above 4 properties.

1.2 Orthogonal functions

Given functions ψ_n, we have the inner product

$$\langle \psi_n, \psi_m \rangle := \int_0^1 \psi_n(x)\psi_m(x)\, dx.$$

We will construct an orthonormal set of functions ψ_n (i.e. $\langle \psi_n, \psi_m \rangle = 0$, $\langle \psi_n, \psi_n \rangle = 1$).[1]

[1] Orthogonal polynomials are very useful in random matrix theory.

Let ψ_n be an orthonormal basis of $L^2([0,1])$. Then if $\psi \in L^2$, we have

$$\phi = \sum_k \langle \phi, \psi_k \rangle \psi_k.$$

Now define

$$W_n(t) = \sum_{k=1}^{n} X_k \cdot \int_0^t \psi_k(s)\, ds, \qquad X_k \sim \text{iid } \mathcal{N}(0,1).$$

Proposition 1.1. *For every $t \in [0,1]$, $W_n(t)$ is Gaussian with $\mathbb{E}[W_n(t)] = 0$ and variance $\mathbb{E}[W_n^2(t)] \xrightarrow{n\to\infty} t$.*

Proof. For the variance,

$$\begin{aligned}
\mathbb{E}[W_n(t)^2] &= \sum_{k=1}^{n} \left| \int_0^t \psi_k(s)\, ds \right|^2 \\
&= \sum_{k=1}^{n} \left| \left\langle \mathbb{1}_{[0,t]}, \psi_k \right\rangle \right| \\
&\xrightarrow{n\to\infty} \left\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,t]} \right\rangle \\
&= t.
\end{aligned}$$

□

Furthermore, $W_n(t) - W_n(s) \xrightarrow{d} \mathcal{N}(0, t-s)$, and $(W_n(I_1), W_n(I_2)) \xrightarrow{d} (B(I_1), B(I_2))$, where B is a Brownian motion.

1.3 Haar functions

We want to say that $W_n(\cdot) \xrightarrow{d} B(\cdot)$. But the limit of the left hand side will be an L^2 funciton, not necessarily a continuous function. Can we find some orthonormal basis of polynomials such that $W_n(\cdot)$ has a $\|\cdot\|_\infty$ limit in $C([0,1])$? Recall the following result:

Proposition 1.2. *Let $f_n \in C([0,1])$ for each n. If $f_n \to f$ uniformly, then $f \in C([0,1])$.*

Now we only need to find an orthonormal basis such that $W_n(\cdot)$ are a Cauchy sequence in $C([0,1])$.

Definition 1.1. The **Haar functions** are the functions

$$f_k^n(x) = \begin{cases} 2^{n/2} & x \in [2^{-n}k, 2^{-n}k + 2^{-(n+1)}] \\ -2^{n/2} & x \in (2^{-n}k + 2^{-(n+1)}, 2^{-n}(k+1)]. \end{cases}$$

Now define $\psi_{2^n+k} = f_k^n$, and let $\widetilde{W}_n = W_{2^n}$. Then

$$\|\widetilde{W}_{n+1} - \widetilde{W}_n\| \leq 2^{-n/2} \max\{|X_{2n}|, \ldots, |X_{2^{n+1}}|\}.$$

We have that $\mathbb{P}(X_i > n) \sim e^{-n^2/2}$, so

$$\mathbb{P}(\max\{x_1, \dots, X_{2^n}\}) \lesssim e^{-n^2/4}$$

for large n. So

$$\|\widetilde{W}_{n+1} - \widetilde{W}\|_\infty \leq 2^{-n+1}n$$

with probability $1 - e^{-n^2/4}$. Using the Borel-Cantelli lemma, we get that the probability these events don't hold infinitely often equals 0.

This method is good for calculating things such as the distribution of the last zero of Brownian motion in $[0, 1]$.